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Analytical and Numerical Aspects of Certain Nonlinear Evolution Equations. III. Numerical, Korteweg–de Vries Equation

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Various numerical methods are used in order to approximate the Korteweg–de Vries equation, namely: (i) Zabusky–Kruskal scheme, (ii) hopscotch method, (iii) a scheme due to Goda, (iv) a proposed local scheme, (v) a proposed global scheme, (vi) a scheme suggested by Kruskal, (vii) split step Fourier method by Tappert, (viii) an improved split step Fourier method, and (ix) pseudospectral method by Fornberg and Whitham. Comparisons between our proposed scheme, which is developed using notions of the inverse scattering transform, and the other utilized schemes are obtained.

1. INTRODUCTION

The Korteweg–de Vries equation (KdV) introduced in [1] was originally derived in order to describe the behavior of one-dimensional shallow water waves with small but finite amplitudes. More recently, this equation also has been found to describe wave phenomena in plasma physics [2, 3], anharmonic crystals [4, 5] bubble-liquid mixtures [6, 7], etc. There has been great interest in this equation because of its special properties. A substantial review of this work can be found in [8, 9]. Zabusky and Kruskal [10] discovered the concept of solitons localized waves with special interaction properties, while studying the results of a numerical computation (describing an anharmonic lattice) on the KdV equation. This motivated the work of Gardner *et al.* [11] and led to the explosion of both the theoretical and numerical work which is still growing today. Many analytical results are available for equations which exhibit exact multisoliton behavior, when an associated scattering problem can be found. Of course there are many examples of inexact, or quasi-soliton behavior. For these problems little or no analytical results are known and numerical studies are essential in order to develop an understanding of the phenomena. This work aims to compare a proposed scheme which was developed in paper I using notions of the

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inverse scattering transform (IST) and certain other known numerical methods for the KdV equation

$$u_t + 6uu_x + u_{xxx} = 0. \tag{1.1}$$

These results can be extended to cover many other related equations as well.

The following numerical methods are applied to the KdV equation:

1. Finite difference methods.
   (a) Explicit methods.
      (i) Zabusky and Kruskal scheme [10, 12].
   (b) Implicit methods.
      (i) Hopscotch method [13].
      (ii) A scheme due to Goda [14].
      (iii) The proposed scheme [15].
      (iv) A scheme suggested by M. Kruskal [16].
2. Finite Fourier transform or pseudospectral methods.
      (i) Split step Fourier method introduced by Tappert [17].
      (ii) Pseudospectral method introduced by Fornberg and Whitham [18].

As in paper II in order to compare schemes, our approach for comparison is to (a) fix the accuracy (L_∞) for computations beginning at $t = 0$ and ending at $t = T$; (b) leave other parameters free (e.g., Δt, or Δx), and compare the computing time required to attain such accuracy for various choices of the parameters.

Various methods are applied to the KdV equation (1.1) subject to the following conditions:

(a) *The Initial Conditions*

(i) 1-*Soliton Solution*

The exact solution of (1.1) on the infinite interval is

$$u(x, t) = A \operatorname{sech}^2(kx - \omega t - \eta_0), \tag{1.2}$$

where

$$\omega = 4k^2, \qquad A = 2k^2, \qquad \eta_0 = \text{constant}.$$

For initial conditions, Eq. (1.2) is used at $t = 0$, and different values of A are tested and η_0 is chosen to be zero.

(ii) *Collisions of Two Solitons*

The exact solution of (1.1) on the infinite interval is

$$u(x, t) = 2(\log f)_{xx}, \tag{1.3}$$

where

$$f = 1 + e^{\eta_1} + e^{\eta_2} + e^{\eta_1+\eta_2+A_{12}},$$

$$\eta_i = k_i x - k_i^3 t + \eta_i^{(0)},$$

and

$$e^{A_{ij}} = \left(\frac{k_i - k_j}{k_i + k_j}\right)^2.$$

For initial conditions, Eq. (1.3) is used at $t = 0$, and two different sets of values of the parameters are studied, namely,

$$k_1 = 1, \qquad k_2 = \sqrt{2}, \qquad \eta_1^{(0)} = 0, \qquad \eta_2^{(0)} = 2\sqrt{2},$$

and

$$k_1 = 1, \qquad k_2 = \sqrt{5}, \qquad \eta_1^{(0)} = 0, \qquad \eta_2^{(0)} = 10.73,$$

and the solitons are allowed to interact and return to their original shapes.

(a) *The Boundary Conditions*

Periodic boundary conditions on the interval $[-20, 20]$ are imposed.

The numerical solution is compared with the exact solution. In addition, two of the conserved quantities are computed, namely; $\int u^2 dx$, and $\int |2u^3 - (u_x)^2| \, dx$.

2. The Representation of the KdV Equation Using Numerical Methods

1. *Finite Difference Methods*

(i) *Zabusky and Kruskal Scheme*

In their original work, Zabusky and Kruskal [10], used the explicit leapfrog finite difference scheme

$$u_n^{m+1} = u_n^{m-1} - 2\frac{\Delta t}{\Delta x}(u_{n+1}^m + u_n^m + u_{n-1}^m)(u_{n+1}^m - u_{n-1}^m)$$

$$- \frac{\Delta t}{(\Delta x)^3}(u_{n+2}^m - 2u_{n+1}^m + 2u_{n-1}^m - u_{n-2}^m), \tag{2.1}$$

where $u_n^m = u(n\Delta x, m\Delta t)$; n and m are integers. This scheme is consistent with Eq. (1.1) and the truncation error is of order $(O((\Delta t)^2) + O((\Delta x)^2))$. The linear stability requirement for this scheme is

$$\frac{\Delta t}{\Delta x}\left|-2u_0 + \frac{1}{(\Delta x)^2}\right| \leqslant \frac{2}{3\sqrt{3}} \tag{2.2}$$

(where u_0 is the maximum value of u in the range of interest). This means that a very small time step must be used to preserve stability. For the initial time step one may use the uncentered scheme

$$u_n^1 = u_n^0 - \frac{\Delta t}{\Delta x}(u_{n+1}^0 + u_n^0 + u_{n-1}^0)(u_{n+1}^0 - u_{n-1}^0)$$

$$- \frac{1}{2(\Delta x)^3}(u_{n+2}^0 - 2u_{n+1}^0 + 2u_{n-1}^0 - u_{n-2}^0). \tag{2.3}$$

(b) *Implicit Methods*

(i) *Hopscotch Method*

In 1976 Greig and Morris [13] proposed a hopscotch scheme for the KdV equation (1.1).

With $f = u^2/2$, the scheme is

$$u_n^{m+1} = u_n^m - 3\frac{\Delta t}{\Delta x}(f_{n+1}^m - f_{n-1}^m) - \frac{\Delta t}{2(\Delta x)^3}(u_{n+2}^m - 2u_{n+1}^m + 2u_{n-1}^m - u_{n-2}^m), \tag{2.4a}$$

$$u_n^{m+1} = u_n^m - 3\frac{\Delta t}{\Delta x}(f_{n+1}^{m+1} - f_{n-1}^{m+1}) - \frac{\Delta t}{2(\Delta x)^3}(u_{n+2}^{m+1} - 2u_{n+1}^{m+1} + 2u_{n-1}^{m+1} - u_{n-2}^{m+1}). \tag{2.4b}$$

To implement the scheme, we employ (2.4a) for those grid points for which $(n+m)$ is even and (2.4b) for those for which $(n+m)$ is odd. A quasi-tridiagonal system of equations must be solved at each time level. An optimization of Gaussian elimination method is used to solve this system (see paper II, Appendix A). The linear stability requirement for this scheme is that [13]

$$\frac{\Delta t}{(\Delta x)^3} \leqslant \left| \frac{1}{(\Delta x)^2 u_0 - 2} \right| \tag{2.5}$$

(u_0 is the maximum value of u in the range of interest). The truncation error of this scheme is of order $(O((\Delta t)^2) + O((\Delta x)^2))$.

(ii) *A Scheme Due to Goda*

This implicit scheme for approximating the KdV equation (1.1) is given by

$$\frac{1}{\Delta t}(u_n^{m+1} - u_n^m) + \frac{1}{\Delta x}\{u_{n+1}^{m+1}(u_n^m + u_{n+1}^m) - u_{n-1}^{m+1}(u_n^m + u_{n-1}^m)\}$$

$$+ \frac{1}{2(\Delta x)^3}\{u_{n+2}^{m+1} - 2u_{n+1}^{m+1} + 2u_{n-1}^{m+1} - u_{n-2}^{m+1}\} = 0. \tag{2.6}$$

The truncation error of this scheme is $O(\Delta t) + O((\Delta x)^2)$. This scheme is unconditionally stable according to linear analysis. In order to apply this scheme, we have to solve a quasi-pentagonal system of equations at each time level:

$$\begin{bmatrix}
x & x & x & & & & & x & x \\
x & x & x & x & & & & & x \\
x & x & x & x & x & & & & \\
 & x & x & x & x & x & & & \\
 & & x & x & x & x & x & & \\
 & & & & & & & & \\
 & & & & x & x & x & x & x \\
x & & & & & x & x & x & x \\
x & x & & & & & x & x & x
\end{bmatrix}$$

An optimization of Gaussian elimination method is used to solve this system of equations.

(iii) *The Proposed Scheme Which Is Based on the IST (see Paper I)*

First, consider the local scheme with $A^{(0)} = (3/2)(\Delta t/(\Delta x)^3)$, which can be written as

$$\begin{aligned}\frac{u_n^{m+1} - u_n^m}{\Delta t} = {} & \frac{1}{2(\Delta x)^3}\left[u_{n-1}^{m+1} - 3u_n^{m+1} + 3u_{n+1}^{m+1} - u_{n+2}^{m+1} + u_{n-2}^m - 3u_{n-1}^m + 3u_n^m - u_{n+1}^m\right] \\ & - \frac{3}{2\Delta x}\left[(u_n^m)^2 - (u_n^{m+1})^2\right] - \frac{1}{2\Delta x}\left\{u_{n+1}^{m+1}(u_n^{m+1} + u_{n+1}^{m+1} + u_{n+2}^{m+1})\right. \\ & \left. - u_{n-1}^m(u_n^m + u_{n-1}^m + u_{n-2}^m)\right\}. \end{aligned} \tag{2.7}$$

The truncation error of this scheme is $O((\Delta t)^2) + O((\Delta x)^2)$.

To implement this scheme, Eq. (2.7) for the new time level can be written as [16]

$$u_{n+2}^{m+1} - 3u_{n+1}^{m+1} + (3 + \varepsilon)\, u_n^{m+1} - u_{n-1}^{m+1} = B_n. \tag{2.8}$$

where

$$\varepsilon \equiv \frac{2(\Delta x)^3}{\Delta t} \ll 1$$

(Δt is supposed to be of the same order as Δx) and

$$\begin{aligned} B_n = {} & u_{n-2}^m - 3u_{n-1}^m - u_{n+1}^m + (3 + \varepsilon)\, u_n^m \\ & - 3(\Delta x)^2[(u_n^m)^2 - (u_n^{m+1})^2] - (\Delta x)^2[u_{n+1}^{m+1}(u_n^{m+1} + u_{n+1}^{m+1} + u_{n+2}^{m+1}) \\ & - u_{n-1}^m(u_n^m + u_{n-1}^m + u_{n-2}^m)]. \end{aligned} \tag{2.9}$$

This can be solved by a version of the Crank–Nicolson back and forth sweep method for the heat equation [20]. We seek an equation of the form

$$u_{n+1}^{m+1} = au_n^{m+1} + b_n^{m+1}, \tag{2.10}$$

which is suitable for computing u_n^{m+1} explicitly by sweeping to the right. For stability $|a| \leqslant 1$. Repeated substitution into Eq. (2.8) to eliminate u_{n+2}^{m+1}, u_{n+1}^{m+1}, and u_n^{m+1} in favor of u_{n-1}^{m+1} gives

$$b_{n+1}^{m+1} + (a-3)\, b_n^{m+1} + (a^2 - 3a + 3 + \varepsilon)\, b_{n-1}^{m+1} + (a^3 - 3a^2 + 3a + \varepsilon a - 1)\, u_{n-1}^{m+1} = B_n. \tag{2.11}$$

Requiring the u_{n-1}^{m+1} term to drop out determines a (uniquely since $|a| \leqslant 1$) as a solution of

$$(a-1)^3 + \varepsilon a = 0 \tag{2.12}$$

and leaves for b_n (at the new time level) a second-order difference equation. The corresponding homogeneous equation of (2.11) has a solution of the form

$$b_n = k^n \tag{2.13}$$

if the constant k satisfies

$$k^2 + (a-3)\, k + a^2 - 3a + 3 + \varepsilon = 0. \tag{2.14}$$

It can be shown, or verified, that the two roots k of this equation are the two roots with $|k| > 1$ other than that of the cubic equation determining $a < 1$ above. It follows that b can be computed explicitly by sweeping to the left

$$b_{n-1} = (3a - a^2)\, b_n - ab_{n+1} + aB_n \tag{2.15}$$

(Eq. (2.15) is obtained from (2.11) and (2.12)). To obtain the solution u_n, first solve for b_n from (2.15) then use (2.10) to calculate u_n.

In order to implement this sweeping technique, the same iteration method used for the sweeping technique discussed in paper II is used. The only difference is that we have to assume initial values not just for b_n but also for b_{n+1}.

Second, for the global method which can be written as (paper I, Equation $(2.30)_*$)

$$\begin{aligned}
&\frac{S_n^m}{1 - S_n^m}\Bigg\{A_-^{(0)} - \sum_{l=-\infty}^{n}\Bigg[E_{l+1} + S_l^{m+1} W_l (A_-^{(2)} + C_{l-2}) \\
&\quad - \Bigg\{D_-^{(4)} \gamma_{l-1} + D_-^{(2)} + \sum_{k=-\infty}^{l-1} (H_k + G_k)\Bigg\} S_l^{m+1} \gamma_l + (\gamma_l - 1)\Bigg] W_l^{-1}\Bigg\} W_n \\
&\quad - \frac{S_n^{m+1}}{1 - S_n^{m+1}}\Bigg\{D_-^{(0)} + \sum_{l=-\infty}^{n-1}\Bigg[\frac{-S_l^m}{S_{l+1}^{m+1}}\{\gamma_{l+1}^{-1} N_{l+1} - N_l + M_l \\
&\quad + S_{l+1}^{m+1} Z_l - S_{l+1}^m \gamma_{l+1}^{-1} N_{l+1}\} + \gamma_l T_{l-2} + (\gamma_l - 1)\Bigg] W_l^{-1}\Bigg\} W_{n-1} \\
&\quad + \frac{1}{1 - S_n^m} E_{n+1} - \frac{1}{1 - S_n^{m+1}} T_{n-2} = \frac{S_n^{m+1} - S_n^m}{(1 - S_n^{m+1})(1 - S_n^m)},
\end{aligned} \tag{2.16}$$

where

$$E_n = A_-^{(2)} S_n^m W_{n-1} - S_n^{m+1} D_-^{(2)} + H_n + G_n - S_n^{m+1} \sum_{k=-\infty}^{n} (H_k + G_k)$$

$$+ S_n^m W_{n-1} C_{n-1} - S_n^m D_-^{(4)}, \qquad C_n = A_-^{(4)} + \sum_{j=-\infty}^{n} P_j W_j^{-1},$$

$$T_n = \gamma_{n+1} M_n + S_{n+1}^{m+1} \gamma_{n+1} Z_n - S_{n+1}^m N_{n+1}, \qquad M_n = S_n^{m+1} W_n A_-^{(4)} - S_n^m D_-^{(4)},$$

$$Z_n = (A_-^{(2)} + \sum_{j=-\infty}^{n} Q_j W_j^{-1}) W_n, \qquad N_n = D_-^{(2)} + \sum_{j=-\infty}^{n} F_j,$$

$$W_n = \prod_{i=-\infty}^{n} \gamma_i, \quad \gamma_i = \left(\frac{1 - S_i^m}{1 - S_i^{m+1}}\right),$$

$$H_k = A_-^{(4)}(S_{k+1}^m \gamma_k - S_k^m) W_{k-1}, \qquad G_k = (S_k^m - S_{k+1}^{m+1}) D_-^{(4)},$$

$$F_j = A_-^{(4)}(S_j^{m+1} W_j - S_{j-1}^{m+1} W_{j-1}) + D_-^{(4)}(S_{j-1}^m - S_j^{m+1}),$$

$$P_j = A_-^{(4)}(S_j^{m+1} - S_{j+1}^m) W_j + D_-^{(4)}(S_{j+1}^{m+1} - S_j^{m+1} \gamma_j),$$

$$Q_j = (S_{j-1}^{m+1} - S_j^m) W_j A_-^{(4)} - (S_{j-1}^m \gamma_j - S_j^m) D_-^{(4)},$$

$$A_-^{(2)} = -\frac{2}{3} A_-^{(0)} + \frac{1}{2}\alpha, \qquad D_-^{(2)} = -\frac{2}{3} A_-^{(0)} - \frac{1}{2}\alpha,$$

$$A_-^{(4)} = \frac{1}{6} A_-^{(0)} - \frac{1}{4}\alpha, \qquad D_-^{(4)} = \frac{1}{6} A_-^{(0)} + \frac{1}{4}\alpha, \quad \alpha = \frac{\Delta t}{(\Delta x)^3},$$

$A_-^{(0)}$ = arbitrary constant and $S_n^m = 1 - e^{-(\Delta x)^2 u_n^m}$.

With $A_-^{(0)} = \frac{3}{2}\alpha$, the same idea is applied and the only difference is in the B_n term. This proposed scheme is unconditionally stable, and has a truncation error of order $(O((\Delta t)^2 + O((\Delta x)^2))$.

(iv) *A Scheme Suggested by M. Kruskal*

Kruskal [16] has suggested the numerical scheme

$$\frac{u_n^{m+1} - u_n^m}{\Delta t} + \frac{u_{n+2}^{m+1} - 3u_{n+1}^{m+1} + 3u_n^{m+1} - u_{n-1}^{m+1}}{2(\Delta x)^3}$$

$$+ \frac{u_{n+1}^m - 3u_n^m + 3u_{n-1}^m - u_{n-2}^m}{2(\Delta x)^3} = 0 \tag{2.17}$$

for

$$u_t + u_{xxx} = 0. \tag{2.18}$$

Kruskal did not suggest any particular numerical scheme for the nonlinear part of the KdV equation (1.1). The following scheme is used to solve the KdV equation (1.1):

$$\frac{u_n^{m+1}-u_n^m}{\Delta t}+\frac{u_{n+2}^{m+1}-3u_{n+1}^{m+1}+3u_n^{m+1}-u_{n-1}^{m+1}}{2(\Delta x)^3}$$
$$+\frac{u_{n+1}^m-3u_n^m+3u_{n-1}^m-u_{n-2}^m}{2(\Delta x)^3}+3\left\{\frac{\theta}{4\Delta x}[(u^2)_{n+1}^{m+1}-(u^2)_{n-1}^{m+1}\right.$$
$$+(u^2)_{n+1}^m-(u^2)_{n-1}^m]+\frac{1-\theta}{2\Delta x}[u_n^{m+1}(u_{n+1}^{m+1}-u_{n-1}^{m+1})$$
$$\left.+u_n^m(u_{n+1}^m-u_{n-1}^m)]\right\}=0. \tag{2.19}$$

To implement this scheme, the above mentioned sweeping/iteration technique is used. Several values of θ are employed and experimentally we find that $\theta=\frac{2}{3}$ gives the best results.

This scheme is unconditionally stable according to linear stability, and has a truncation error of order $(O((\Delta t)^2)+O((\Delta x)^2))$.

2. *Finite Fourier Transform or Pseudospectral Methods*

(i) *Split Step Fourier Method by F. Tappert*

For convenience the spatial period was normalized to $[0, 2\pi]$, then Eq. (1.1) becomes

$$u_t+6\frac{\pi}{p}uu_X+\frac{\pi^3}{p^3}u_{XXX}=0, \tag{2.20}$$

where p is half the length of the interval of interest, and $X=(x+p)\,\pi/p$.

As discussed in part II of this paper the essence of the solution method is to alternate between two steps: (1) advance the solution using only the nonlinear term by means of a (implicit) finite difference approximation. (2) advance the solution using only the linear term by means of the discrete fast Fourier transform (FFT).

To implement this method for the KdV equation (2.20), as the first step, one first approximates,

$$u_t+6\frac{\pi}{p}uu_X=0. \tag{2.21}$$

A straightforward discretization is

$$\tilde{u}_n^{m+1}=u_n^m-\frac{3}{4}\frac{\pi}{p}\frac{\Delta t}{\Delta X}\{(\tilde{u}^2)_{n+1}^{m+1}-(\tilde{u}^2)_{n-1}^{m+1}+(u^2)_{n+1}^m-(u^2)_{n-1}^m\}, \tag{2.22}$$

where $\tilde{u}$ is a solution of Eq. (2.21) and u is the solution of Eq. (2.20). For the second step, we would take,

$$u(X_j, t + \Delta t) = F^{-1}(e^{ik^3\pi^3/p^3\Delta t}F(\tilde{u}(X_j, t))), \tag{2.23}$$

where F denotes discrete Fourier transform and F^{-1} its inverse. This scheme is second order accurate in time and space (which comes from using Eq. (2.22) to approximate Eq. (2.21)), and unconditionally stable according to linear analysis. In order to find $F(\tilde{u})$ and F^{-1} the FFT technique is used. We have found however that an improved discretization of (2.21) works considerably better. Specifically the truncation error of the split step Fourier method is improved to be of order $(O((\Delta t)^2) + O((\Delta x)^4))$ instead of order $(O((\Delta t^2) + O((\Delta x)^2))$, by approximating Eq. (2.21) according to

$$\tilde{u}_n^{m+1} = u_n^m - \frac{\Delta t}{8\Delta X}\frac{\pi}{p}\{[8(\tilde{u}^2)_{n+1}^{m+1} - 8(\tilde{u}^2)_{n-1}^{m+1} - (\tilde{u}^2)_{n+2}^{m+1} + (\tilde{u}^2)_{n-2}^{m+1}] + [8(u^2)_{n+1}^m - 8(u^2)_{n-1}^m - (u^2)_{n+2}^m + (u^2)_{n-2}^m]\}. \tag{2.24}$$

Also, one may improve the truncation error to be of order $(O(\Delta t)^2 + O(\Delta x)^p)$ for all p, see the pseudospectral method (Fornberg and Whitham) below.

(ii) *Pseudospectral Method by Fornberg and Whitham* [18]

As mentioned in part II of this paper, this is a Fourier method in which $u(x, t)$ is transformed into Fourier space with respect to x. Again for convenience the spatial period is normalized to $[0, 2\pi]$. This interval is discretized by N equidistant points, with spacing $\Delta X = 2\pi/N$. The function $u(X, t)$, numerically defined only on these points, can be transformed to the discrete Fourier space by

$$\hat{u}(k, t) = Fu = \frac{1}{\sqrt{N}}\sum_{j=0}^{N-1} u(j\Delta X, t)\, e^{-2\pi ijk/N},$$
$$k = -\frac{N}{2},\ldots, -1, 0, 1,\ldots, \frac{N}{2} - 1. \tag{2.25}$$

The inversion formula is

$$u(j\Delta X, t) = F^{-1}\hat{u} = \frac{1}{\sqrt{N}}\sum_{k} \hat{u}(k, t)\, e^{2\pi ijk/N},$$
$$k = -\frac{N}{2},\ldots, -1, 0, 1,\ldots, \frac{N}{2} - 1. \tag{2.26}$$

These transforms can be performed, efficiently with the fast Fourier transform algorithm [21]. With this scheme, u_X can be evaluated as $F^{-1}\{ikFu\}$, u_{XXX} as

$F^{-1}\{-ik^3Fu\}$ and so on. Combined with a leap-frog time step the KdV equation (2.20) would then be approximated by

$$u(X, t+\Delta t) - u(X, t-\Delta t) + 2i\frac{6\pi}{p}\Delta t\, u(X, t)\, F^{-1}(kF(u)) - 2i\Delta t\frac{\pi^3}{p^3}F^{-1}(k^3F(u)) = 0. \tag{2.27}$$

Fornberg and Whitham make a modification in the last term, however, and take

$$u(X, t+\Delta t) - u(X, t-\Delta t) + 2i\frac{6\pi}{p}\Delta t\, u(X, t)\, F^{-1}(kF(u)) - 2iF^{-1}\left\{\sin\left(\frac{\pi^3k^3}{p^3}\Delta t\right)F(u)\right\} = 0. \tag{2.28}$$

The difference between Eq. (2.27) and (2.28) is in the approximation of the linear equation

$$u_t + \frac{\pi^3}{p^3}u_{XXX} = 0. \tag{2.29}$$

The linear part of Eq. (2.28) is exactly satisfied for any solution of Eq. (2.29) [18]. Also it turns out that the linearized stability condition is less restrictive for (2.28) than for (2.27): $\Delta t/(\Delta x)^3 < 3/2\pi^3 \simeq 0.1520$ compared to $\Delta t/(\Delta x)^3 < 1/\pi^3 \approx 0.0323$ (for details see [18]).

Since the Fornberg and Whitham scheme is explicit, it is natural to consider Crank–Nicolson type implicit version, e.g.,

$$u(X, t+\Delta t) - u(X, t) + 3i\Delta t\pi/p\{u(X, t+\Delta t)\, F^{-1}(kF(u(X, t+\Delta t))) + u(X, t)\, F^{-1}(kF(u(X, t)))\} - \frac{i\Delta t}{2}\left(\frac{\pi}{p}\right)^3\{F^{-1}(k^3F(u(X, t+\Delta t))) + F^{-1}(k^3F(u(X, t)))\} = 0. \tag{2.30}$$

This scheme (2.30) is unconditionally stable according to linear stability. We make some remarks about (2.30) in the conclusions.

3. Conclusions

Various numerical methods are used in order to approximate the KdV equation (1.1), namely, (i) Zabusky and Kruskal scheme (2.1), (ii) hopscotch method (2.4), (iii) a scheme due to Goda (2.6), (iv) a proposed local scheme (2.7), (v) a proposed global scheme (2.16), (vi) a scheme suggested by Kruskal (2.19), (vii) split step

Fourier method by Tappert (2.22)–(2.24), and (viii) pseudospectral method by Fornberg and Whitham (2.28).

As in part II, our approach for comparison is to (a) fix the accuracy (L_∞) for computations beginning at $t = 0$ and ending at $t = T$; (b) leave other parameters free (e.g., Δt, or Δx), and compare the computing time required to attain such accuracy for various choices of the parameters.

Two sets of initial conditions were studied: (A) 1-soliton solution with different values of the amplitude, (B) collisions of two solitons with different values of the parameters. According to this approach we have made the following conclusions:

(1) The scheme of Goda required a long time compared to the other ((i), (ii), (iv), (vi), (vii), (viii)) schemes.

(2) Zabusky and Kruskal's scheme was good for low amplitudes, but it needed more computing time than the other remaining methods ((ii), (iv), (vi), (vii), (viii)) for high amplitudes.

(3) The calculations for the previous two methods; Goda, and Zabusky–Kruskal, were not carried out for the 1-soliton case with amplitude = 4. They needed very long computing time.

(4) The Tappert and hopscotch schemes took less computing time than the previous two schemes. For small amplitudes hopscotch was more efficient than Tappert, and they behave almost the same for fairly small amplitudes. On the other hand for relatively high amplitudes the Tappert scheme turned out to be better.

(5) The suggested scheme by Kruskal is in general faster than the previous schemes ((i), (ii), (iii), (v), (vii)).

(6) The Fornberg and Whitham method is much faster than the suggested scheme by Kruskal, it is roughly three times faster for small amplitudes and six times faster for high amplitudes. Also (2.27) was tried, but (2.28) proved to be somewhat faster. In addition, the implicit version (2.30) was implemented and did not prove to be faster than (2.28).

(7) The proposed local scheme is the best amongst all the utilized schemes. It was roughly eight times faster than the suggested scheme by Kruskal. (See remark below Eq. (2.18)). This certainly shows that the approximation of the nonlinear term is crucial. Also, it was roughly one and a half times faster than the Fornberg and Whitham scheme. This despite the fact that the local scheme is only $O((\Delta t)^2, (\Delta x)^2)$ whereas Fornberg and Whitham method is of order $O((\Delta t)^2, (\Delta x)^p)$ for all p. These results suggest that "IST generated" schemes will be good approximations for equations which are in fact exactly solvable by the IST. The proposed global scheme was implemented and proved to be faster than some of the utilized schemes, but much slower than its local version. Since the global scheme is so complicated and cumbersome, we are not convinced that our methods of implementation were optimal. Optimizing the implementation of our global scheme will be under further investigation in the near future. (The following tables and figures exhibit the results). All the numerical calculations were inspected at every step by using the conserved

TABLE I

Comparison of the Computing Time (E) Which Is Required to Attain an Accuracy (L_∞) < 0.005 for Computations Beginning at $t = 0$ and Ending at $t = 1.0$, for the Numerical Methods Utilized in Solving the KdV Equation[a]

No.	Method	Mesh size	Time		$t = T$	L_∞	v_1	v_2	Normalized E
			min	sec					
1.	Explicit (Zabusky and Kruskal)	$\Delta x = 0.1739$ $\Delta t = 0.002$	E: 0	28	0.25	0.00173	0.00000	0.00479	
					0.5	0.00283	0.00001	0.00480	4
					1.0	0.00469	0.00000	0.00478	
2.	Goda	$\Delta x = 0.1$ $\Delta t = 0.002$	E: 4	4	0.25	0.00134	−0.00080	0.00026	
					0.5	0.00245	−0.00160	−0.00107	34.9
					1.0	0.00492	−0.00318	−0.00372	
3.	Hopscotch	$\Delta x = 0.2$ $\Delta t = 0.003$	E: 0	23	0.25	0.00178	−0.00001	0.00627	
					0.5	0.00292	0.00000	0.00628	4.3
					1.0	0.00472	0.00002	0.00619	
4.	Tappert	$\Delta x = 0.3125$ $\Delta t = 0.004$	E: 1	3	0.25	0.00338	0.00000	−0.00004	
					0.5	0.00401	−0.00001	−0.00004	9
					1.0	0.00494	−0.00001	−0.00005	

5.	Kruskal	$\Delta x = 0.08$			0.25	0.00202	0.00001	0.00101	
		$\Delta t = 0.04$	E: 0	24	0.5	0.00303	0.00001	0.00101	3.4
					1.0	0.00453	0.00002	0.00102	
6.	The proposed	$\Delta x = 0.16$			0.25	0.00146	0.00005	0.00413	
	local scheme	$\Delta t = 0.125$	E: 0	7	0.5	0.00162	0.00007	0.00419	1
					1.0	0.00173	0.00011	0.00426	
7.	The proposed	$\Delta x = 0.13$			0.25	0.00138	0.00000	0.00264	
	global scheme	$\Delta t = 0.12$	E: 0	38	0.5	0.00252	0.00000	0.00263	5.4
					1.0	0.00477	0.00001	0.00263	
8.	Pseudospectral	$\Delta x = 0.625$			0.25	0.00157	0.00225	0.02315	
	by Fornberg and	$\Delta t = 0.0096$	E: 0	12	0.5	0.00162	−0.00112	−0.01090	1.7
	Whitham				1.0	0.00113	0.00185	0.01977	

[a] Two conserved quantities are shown. 1-soliton as an initial condition with amplitude = 1 on the interval $[-20, 20]$.

Δx = The increment in x.

Δt = The increment in t.

$v_1 = (u_1 - u_{10})/u_{10}$, u_{10} = The exact value of $\int u^2\,dx$.

$v_2 = (u_2 - u_{20})/u_{20}$, u_{20} = The exact value of $\int [2u^3 - (u_x)^2]\,dx$.

u_1 = The calculated value of the conserved quantity of the KdV equation which is $\int u^2\,dx$.

u_2 = The calculated value of the conserved quantity of the KdV equation which is $\int [2u^3 - (u_x)^2]\,dx$.

$L_\infty = \max |\tilde{u}_n^m - u_n^m|$, $\tilde{u}_n^m$ is the numerical solution and u_n^m is the exact solution at the point $(\Delta x \cdot n, \Delta t \cdot m)$ for all n, m.

TABLE II

Comparison of the Computing Time (E) Which Is Required to Attain an Accuracy (L_∞) < 0.01 for Computations Beginning at $t = 0$ and Ending at $t = 1.0$, for the Numerical Methods Utilized in Solving the KdV Equation[a]

No.	Method	Mesh size	Time min	Time sec	$t = T$	L_∞	v_1	v_2	Normalized E
1.	Explicit (Zabusky and Kruskal)	$\Delta x = 0.08$ $\Delta t = 0.00019$	E: 9	51	0.25	0.00307	0.00000	0.00203	
					0.5	0.00520	0.00000	0.00203	25.7
					1.0	0.00930	0.00000	0.00203	
2.	Goda	$\Delta x = 0.04$ $\Delta t = 0.00025$	E: 59	28	0.25	0.00236	−0.00080	−0.00082	
					0.5	0.00509	0.00028	−0.00215	155.1
					1.0	0.01282	−0.00318	−0.00480	
3.	Hopscotch	$\Delta x = 0.1$ $\Delta t = 0.0005$	E: 4	32	0.25	0.00371	0.00000	0.00318	
					0.5	0.00592	0.00001	0.00313	11.8
					1.0	0.00994	0.00001	0.00316	
4.	Tappert	$\Delta x = 0.156$ $\Delta t = 0.002$	E: 4	31	0.25	0.00729	−0.00001	−0.00007	
					0.5	0.00820	−0.00003	−0.00009	11.8
					1.0	0.00943	−0.00005	−0.00014	
5.	Kruskal	$\Delta x = 0.04$ $\Delta t = 0.011$	E: 2	43	0.25	0.00418	−0.00001	0.00051	
					0.5	0.00607	0.00001	0.00052	7.1
					1.0	0.00952	0.00002	0.00053	
6.	The proposed local scheme	$\Delta x = 0.1$ $\Delta t = 0.1$	E: 0	23	0.25	0.00237	0.00009	0.00331	
					0.5	0.00246	0.00010	0.00330	1
					1.0	0.00332	0.00014	0.00337	
7.	The proposed global scheme	$\Delta x = 0.05$ $\Delta t = 0.025$	E: 6	7	0.25	0.00228	0.00003	0.00083	
					0.5	0.00443	0.00005	0.00087	16
					1.0	0.00882	0.00008	0.00092	
8.	Pseudospectral by Fornberg and Whitham	$\Delta x = 0.3125$ $\Delta t = 0.0042$	E: 0	40	0.25	0.00299	0.00009	0.00168	
					0.5	0.00323	−0.00003	−0.00074	1.7
					1.0	0.00474	0.00008	0.00162	

[a] Two conserved quantities are shown. 1-soliton as an initial condition with amplitude = 2 on the interval $|-20, 20|$.

TABLE III

Comparison of the Computing Time (E) Which Is Required to Attain an Accuracy (L_∞) < 0.022 for Computations Beginning at $t = 0$ and Ending at $t = 1.0$, for the Numerical Methods Utilized in Solving the KdV Equation[a]

No.	Method	Mesh size	Time min	Time sec	$t = T$	L_∞	v_1	v_2	Normalized E
1.	Hopscotch	$\Delta x = 0.05$			0.25	0.00722	0.00000	0.00158	
		$\Delta t = 6.2 \times 10^{-5}$	E: 77	39	0.5	0.01240	0.00000	0.00157	33.3
					1.0	0.02259	0.00000	0.00158	
2.	Tappert	$\Delta x = 0.073$			0.25	0.016242	0.00000	−0.00001	
		$\Delta t = 0.0008$	E: 26	30	0.5	0.017140	0.00000	−0.00004	11.4
					1.0	0.021586	0.00000	−0.00000	
3.	Kruskal	$\Delta x = 0.03$			0.25	0.00580	−0.00004	0.00050	
		$\Delta t = 0.01$	E: 22	8	0.5	0.01069	−0.00008	0.00044	9.5
					1.0	0.02110	−0.00018	0.00031	
4.	The proposed local scheme	$\Delta x = 0.05$			0.25	0.00992	−0.00002	0.00154	
		$\Delta t = 0.0275$	E: 2	20	0.5	0.01272	−0.00003	0.00152	1
					1.0	0.01747	−0.00004	0.00149	
5.	The proposed global scheme	$\Delta x = 0.025$			0.25	0.00530	0.00001	0.00042	
		$\Delta t = 0.005$	E: 53	23	0.5	0.01072	0.00009	0.00055	22.9
					1.0	0.02163	−0.00006	0.00029	
6.	Pseudospectral by Fornberg and Whitham	$\Delta x = 0.3125$			0.25	0.00600	−0.00067	−0.00653	
		$\Delta t = 0.00115$	E: 3	14	0.5	0.01095	0.00221	0.02220	1.4
					1.0	0.01752	−0.00088	−0.01001	

[a] Two conserved quantities are shown. 1-soliton as an initial condition with amplitude = 4 on the interval $[-20, 20]$.

TABLE IV

Comparison of the Computing Time (E) Which Is Required to Attain an Accuracy (L_∞) < 0.002 for Computations Beginning at $t = 0$ and Ending at $t = 3.0$, for the Numerical Methods Utilized in Solving the KdV Equation[a]

No.	Method	Mesh size	Time min	Time sec	$t = T$	L_∞	v_1	v_2	Normalized E
1.	Explicit (Zabusky and Kruskal)	$\Delta x = 0.12$ $\Delta t = 0.00066$	E: 5	49	0.1	0.00047	0.00000	0.00172	
					0.5	0.00115	0.00000	0.00148	
					1.0	0.00159	0.00000	0.00110	18.4
					2.0	0.00191	0.00000	0.00035	
					3.0	0.00165	0.00000	0.00033	
2.	Goda	$\Delta x = 0.1$ $\Delta t = 0.0005$	E: 46	4	0.1	0.00032	−0.00005	0.00111	
					0.5	0.00105	−0.00022	0.00062	
					1.0	0.00161	−0.00041	0.00001	145.5
					2.0	0.00200	−0.00066	−0.00095	
					3.0	0.00165	−0.00082	−0.00121	
3.	Hopscotch	$\Delta x = 0.13$ $\Delta t = 0.001$	E: 4	43	0.1	0.00032	−0.00005	0.00111	
					0.5	0.00107	−0.00022	0.00062	
					1.0	0.00147	−0.00041	0.00001	14.9
					2.0	0.00191	−0.00066	−0.00095	
					3.0	0.00142	−0.00082	−0.00121	
4.	Tappert	$\Delta x = 0.15625$ $\Delta t - 0.005$	E: 5	22	0.1	0.00171	0.00000	−0.00002	
					0.5	0.00188	0.00000	−0.00007	
					1.0	0.00160	0.00000	−0.00010	16.9
					2.0	0.00126	0.00000	−0.00004	
					3.0	0.00186	0.00000	0.00018	

5.	Kruskal scheme	$\Delta x = 0.08$			0.1	0.00021	0.00000	0.00076	
		$\Delta t = 0.015$	E: 1	46	0.5	0.00088	−0.00001	0.00020	
					1.0	0.00139	0.00000	0.00042	5.6
					2.0	0.00176	0.00003	0.00071	
					3.0	0.00145	0.00002	0.00053	
6.	The proposed	$\Delta x = 0.1$			0.1	0.00080	−0.00001	0.00119	
	local scheme	$\Delta t = 0.14$	E: 0	19	0.5	0.00113	−0.00004	0.00104	
					1.0	0.00135	−0.00018	0.00062	1
					2.0	0.00138	−0.00043	−0.00018	
					3.0	0.00148	−0.00044	−0.00023	
7.	The proposed	$\Delta x = 0.07$			0.1	0.00025	0.00001	0.00060	
	global scheme	$\Delta t = 0.06$	E: 4	21	0.5	0.00051	0.00004	0.00060	
					1.0	0.00080	0.00007	0.00056	13.7
					2.0	0.00115	0.00000	0.00026	
					3.0	0.00204	0.00029	0.00071	
8.	Pseudospectral	$\Delta x = 0.625$			0.1	0.00127	0.00000	0.01517	
	by Fornberg and	$\Delta t = 0.0148$	E: 0	24	0.5	0.00101	−0.00069	−0.00871	
	Whitham				1.0	0.00166	−0.00047	−0.00649	1.3
					2.0	0.00101	−0.00005	0.00033	
					3.0	0.00107	0.00018	0.00272	

[a] Two conserved quantities are shown. Two solitons as an initial condition with amplitudes $\frac{1}{2}$ and 1 respectively, and they are allowed to interact, on the interval $[-20, 20]$.

TABLE V

Comparison of the Computing Time (E) Which Is Required to Attain an Accuracy (L_∞) < 0.02 for Computations Beginning at $t = 0$ and Ending at $t = 2.4$, for the Numerical Methods Utilized in Solving the KdV Equation[a]

No.	Method	Mesh size	Time min	Time sec	$t = T$	L_∞	v_1	v_2	Normalized E
1.	Explicit (Zabusky and Kruskal)	$\Delta x = 0.06$ $\Delta t = 0.000082$	E: 74	5	0.1	0.00177	0.00000	0.00138	
					0.6	0.00655	0.00000	0.00114	
					1.2	0.00842	0.00000	0.00001	62.6
					2.0	0.01734	0.00000	0.00106	
					2.4	0.02220	0.00000	0.00134	
2.	Goda	This method takes very long time to attain the required accuracy, therefore there is no need to carry out the calculations.							
3.	Hopscotch	$\Delta x = 0.0725$ $\Delta t = 0.00019$	E: 32	29	0.1	0.00196	0.00001	0.00201	
					0.6	0.00649	0.00011	0.00181	
					1.2	0.00919	0.00054	0.00085	27.5
					2.0	0.01586	0.00013	0.00174	
					2.4	0.02013	0.00002	0.00196	
4.	Tappert	$\Delta x = 0.15625$ $\Delta t = 0.002$	E: 10	48	0.1	0.01232	0.00000	−0.00010	
					0.6	0.01320	0.00001	−0.00025	
					1.2	0.00859	0.00005	−0.00047	9.1
					2.0	0.01802	−0.00004	−0.00003	
					2.4	0.02086	−0.00006	−0.00013	

5.	Kruskal scheme	$\Delta x = 0.04$			0.1	0.00221	0.00000	0.00060	
		$\Delta t = 0.0045$	E: 11	56	0.6	0.00696	0.00001	0.00048	
					1.2	0.00842	0.00015	0.00001	10.1
					2.0	0.01534	0.00000	0.00041	
					2.4	0.02019	−0.00004	0.00050	
6.	The proposed	$\Delta x = 0.075$			0.1	0.00229	0.00001	0.00216	
	local scheme	$\Delta t = 0.055$	E: 1	11	0.6	0.00321	−0.00026	0.00127	
					1.2	0.01023	−0.00134	0.00014	1
					2.0	0.01613	−0.00045	0.00073	
					2.4	0.01502	−0.00025	0.00161	
7.	The proposed	$\Delta x = 0.035$			0.1	0.00092	0.00001	0.00048	
	global scheme	$\Delta t = 0.0125$	E: 35	50	0.6	0.00478	0.00012	0.00061	
					1.2	0.00662	−0.00018	−0.00023	30.3
					2.0	0.01670	0.00013	0.00058	
					2.4	0.02002	−0.00023	0.00007	
8.	Pseudospectral	$\Delta x = 0.3125$			0.1	0.00454	−0.00004	−0.00317	
	by Fornberg and	$\Delta t = 0.004$	E: 1	12	0.6	0.00791	−0.00028	−0.00367	
	Whitham				1.2	0.00783	0.00033	0.00060	1
					2.0	0.01311	0.00008	0.00153	
					2.4	0.01705	−0.00162	−0.01211	

[a] Two conserved quantities are shown. Two solitons as an initial condition with amplitudes 1/2 and 5/2 respectively, and they are allowed to interact, on the interval $[-20, 20]$.

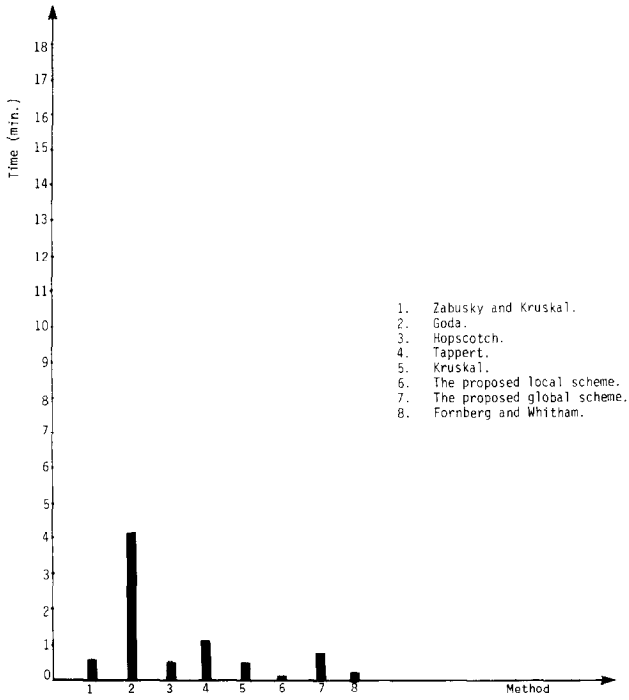

FIG. 1. Displays the computing time (E) which is required by each utilized method given in Table I.

1-soliton, amplitude = 1.

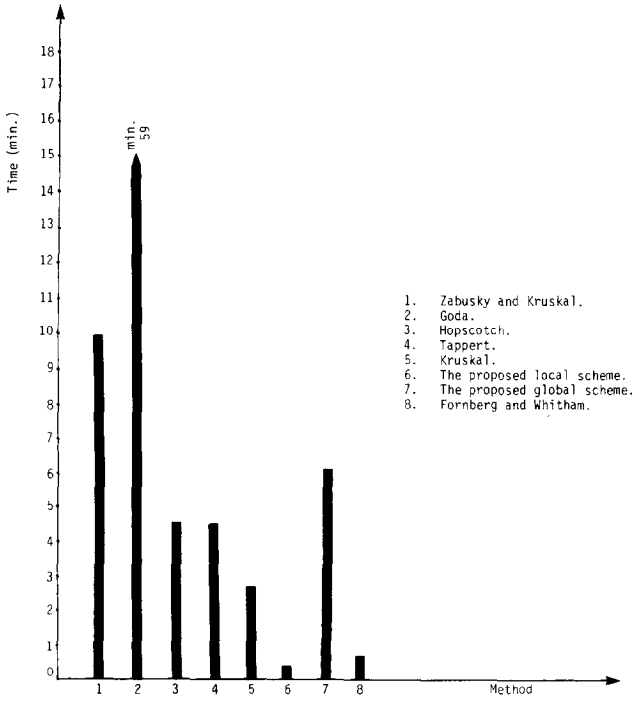

FIG. 2. Displays the computing time (E) which is required by each utilized method given in Table II.

1-soliton, amplitude = 2.

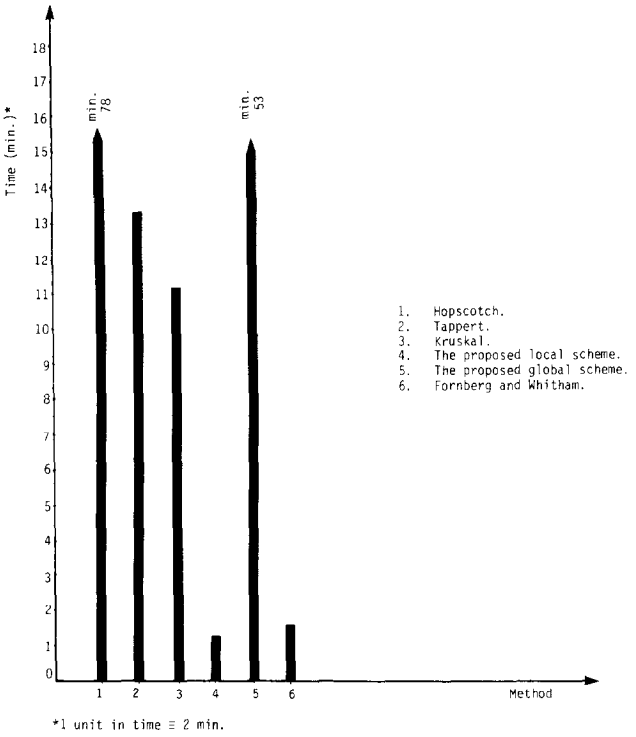

FIG. 3. Displays the computing time (E) which is required by each utilized method given in Table III.

1-soliton, amplitude = 4.

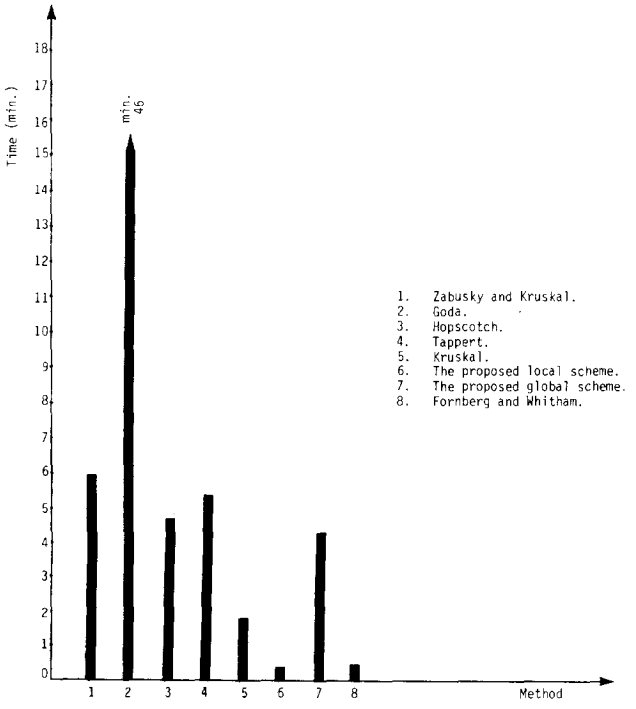

FIG. 4. Displays the computing time (E) which is required by each utilized method given in Table IV.

Two solitons with amplitudes 0.5 and 1.

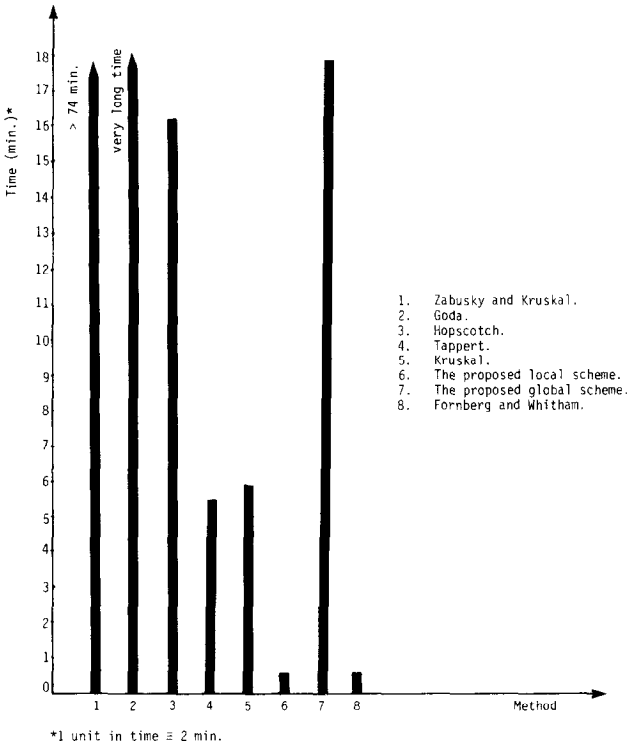

FIG. 5. Displays the computing time (E) which is required by each utilized method given in Table V.

Two solitons with amplitudes 0.5 and 2.5.

quantities $\int u^2\,dx$, and $\int (2u^3 - (u_x)^2)\,dx$. (Table I–V). The two conserved quantities were calculated by means of Simpson's rule. In the finite difference schemes we have discritized u_x using a central difference approximation. In the Fourier methods the derivatives are calculated using Fourier method. The proposed global scheme is the only utilized scheme which has an infinite number of conserved quantities, and true soliton solutions. It is worth mentioning that we calculate the L_2 error norm and find it reflects the same conclusions as the L_∞ norm.

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REFERENCES

1. D. J. KORTEWEG AND G. DEVRIES, On the Change of Form of Long Waves Advancing in a Rectangular Canal, and on a New Type of Long Stationary Wave, *Philos. Mag.* **39** (1895), 422–443.
2. C. S. GARDNER AND G. K. MORIKAWA, "Similarity in the Asymptotic Behavior of Collision-Free Hydromagnetic Waves and Water Waves," New York Univ., Courant Inst. Math. Sci., Res. Report NYO-9082, 1960.
3. H. WASHIMI AND T. TANIUTI, Propagation of ion-acoustic solitary waves of small amplitudes, *Phys. Rev. Lett.* **17** (1966), 996–998.
4. (a) N. ZABUSKY, "A Synergetic Approach to Problems of Nonlinear Wave Propagation and Interaction in Nonlinear Partial Differential Equations" (W. F. Ames, Ed.), pp. 223–258, New York, 1967. (b) N. ZABUSKY, Computational synergetics and mathematical innovation, *J. Comp. Phys.* **43** (1981), 195–249.
5. M. D. KRUSKAL, Asymptotology in numerical computation: progress and plans on the Fermi–Pasta–Ulam problem, *in*, "Proceedings, IBM Scientific Computing Symposium on Large-Scale Problems in Physics," 1965.
6. L. VAN WIJNGAARDEN, "Linear and Non-Linear Dispersion of Pressure Pulses in Liquid-Bubble Mixtures," 6th Symposium on Naval Hydrodynamics, Washington, D. C., 1966.
7. L. VAN WIJNGAARDEN, On the equations of motion for mixtures of liquid and gas bubbles, *J. Fluid Mech.* **33** (1968), 465–474.
8. V. E. ZAKHAROV, S. V. MANAKOV, S. P. NOVIKOV, AND S. P. PITAYEVSKY, "Theory of Solitons," Moscow, Nauk, 1980.
9. M. ABLOWITZ AND H. SEGUR, Solitons and the inverse scattering transform, *SIAM*, Philadelphia, 1981.
10. N. J. ZABUSKY AND M. D. KRUSKAL, Interaction of "solitons" in a collisionless plasma and the recurrence of initial states, *Phys. Rev. Lett.* **15** (1965), 240–243.
11. C. GARDNER, J. GREENE, M. KRUSKAL, AND R. MIURA, *Phys. Rev. Lett.* **19** (1967), 1095.
12. A. C. VLIEGENTHART, *J. Engrg. Math.* **5** (1971), 137.
13. I. S. GREIG AND J. LL. MORRIS, *J. Comp. Phys.* **20** (1976), 60–84.
14. K. GODA, On instability of some finite difference schemes for the Korteweg–deVries equation, *J. Phys. Soc. Japan* **1** (1975).
15. T. TAHA, Ph.D. thesis, Clarkson College, Potsdam, New York, 1982.
16. M. D. KRUSKAL, private communication, 1981.
17. F. TAPPERT, *Lect. Appl. Math. Am. Math. Soc.* **15** (1974), 215–216.
18. B. FORNBERG AND G. B. WHITHAM, *Phil. Trans. Roy. Soc.* **289** (1978), 373.
19. B. FORNBERG, *Math. Comp.* **27** (1973), 45–57.
20. R. D. RICHTMYER AND K. W. MORTON, "Difference Methods for Initial Value Problems," 2nd ed. Wiley–Interscience, New York, 1967.
21. J. W. COOLEY, P. A. W. LEWIS, AND P. D. WELCH, *IEEE Trans. Educ. E-12* **1** (1969), 27–34.
22. G. D. SMITH, "Numerical Solution of Partial Differential Equations," Oxford Univ. Press, New York, 1965.